Inequality-Constrained-Model Selection for the Direct Comparison of Competing Leadership Theories

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Outline

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Preliminaries: Problem

Perceived Problem

- Parameter specification is often rigid
- i.e., parameter is specified to be part or *not* to be part of a model

Wanted: Specificity

- Most often, researchers have quite specific theories
- Model selection incorporating direction and magnitude parameters
- Incorporation inequality constraints in model

Model Selection

To select, in settings in which such is deemed appropriate, from a set of *a priori* specified models, the one model that best balances model *fit* and model *complexity*

Preliminaries: Motivating Example

Consider

- Analysis of variance (ANOVA) type setting
- μ_j denotes the mean of group j, $j = 1, \dots, J$
- One may have a specific theory, say:

$$M_1 : \mu_1 > \mu_2 > \cdots > \mu_J$$

However

Rejection of the null-alternative in the traditional null-hypothesis setting

$$H_0 : \text{all } \mu_j \text{ are equal} \quad vs$$
$$H_a : \text{all } \mu_j \text{ are not equal},$$

does not provide any information on M_1

Preliminaries: Goal

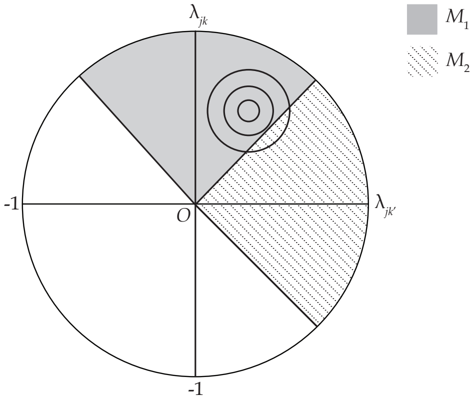
Goal

To develop a generic framework for inequality-constrained-model selection that allows for

i more specificity in model-translations of substantive theory

ii the direct testing of competing theories

Preliminaries: Geometric Feel Solution



- M_1: $\lambda_{jk} > |\lambda_{jk'}|$
- M_2: $\lambda_{jk'} > |\lambda_{jk}|$
- $B_{10} = (7/8) \div (1/4) = 7/2$
- $B_{20} = (1/8) \div (1/4) = 1/2$
- $B_{12} = (7/2) \div (1/2) = 7$

Outline

Example: Data

Background

- Leadership argued to be a key factor influencing employee behavior
- Exemplary leadership particularly important for police bureaucracy

Setting

- Changing landscape: Traditional moral-paternal styles of policing are being amended with more output-steering organizational styles
- Lasthuizen (2008) devised questionnaire operationalizing scales of organizational and ethical leadership styles
- Questionnaire conducted in a Dutch regional police force
- Received responses from 536 police officers *not* in a supervisory position

Example: Data

Validation Questionnaire

A validation by Lasthuizen (2008) led to the extraction of 6 latent factors:

- Inspirational leadership (IL)
- Result-oriented leadership (ROL)
- Passive leadership (PL)
- Role-modeling leadership (RML)
- Integrity-focused leadership (IFL)
- Unethical leadership (UL)

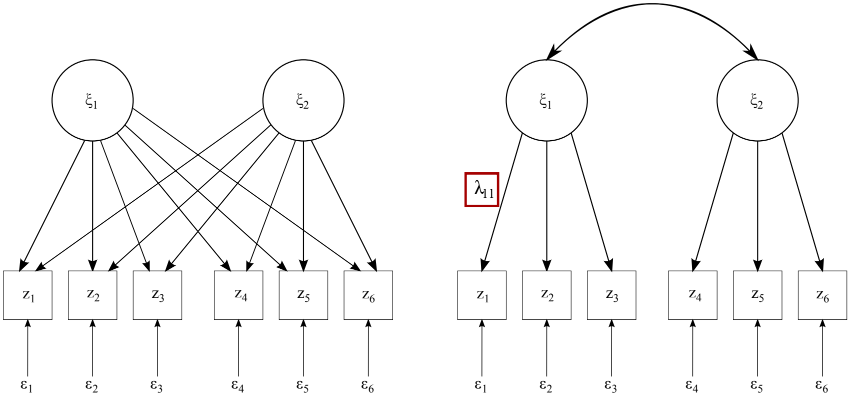
Proposition

Lasthuizen theorizes, the existence of two higher-order latent factors: organizational and ethical leadership styles

Our Aim

- To assess the factor structure of the six leadership scales
- To directly compare competing leadership theories

Example: Model



Example: Undesirable Classical Approach

Classical Approach

In a classical CFA the ponderings by Lasthuizen (2008) are expressed as:

$$\mathbf{\Lambda}_c = \begin{bmatrix} \lambda_{11} & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ 0 & \lambda_{42} \\ 0 & \lambda_{52} \\ 0 & \lambda_{62} \end{bmatrix} \begin{matrix} \text{IL} \\ \text{ROL} \\ \text{PL} \\ \text{RML} \\ \text{IFL} \\ \text{UL} \end{matrix} .$$

Situation Not Ideal

1. It implies a loss of information
2. Non-nestedness of model formulations defies classical testing
3. Exclusion restrictions may be problematic
4. Expectations regarding direction and magnitude cannot be expressed

Example: Theory 1

Lasthuizen Theory

$$\boldsymbol{\Lambda}_1 = \begin{bmatrix} \lambda_{11} & > & |\lambda_{12}| \\ \lambda_{21} > 0 & & \lambda_{22} = 0 \\ \lambda_{31} & > & |\lambda_{32}| \\ |\lambda_{41}| & < & \lambda_{42} \\ |\lambda_{51}| & < & \lambda_{52} \\ \lambda_{61} = 0 & & \lambda_{62} > 0 \end{bmatrix} \begin{matrix} \text{IL} \\ \text{ROL} \\ \text{PL} \\ \text{RML} \\ \text{IFL} \\ \text{UL} \end{matrix} .$$

Organizational vs. ethical leadership styles.

Note

A formulation like

$$\lambda_{11} > |\lambda_{12}| \Rightarrow \begin{cases} \lambda_{11} - \lambda_{12} > 0 \\ \lambda_{11} + \lambda_{12} > 0 \end{cases},$$

indicates the belief that (the positive of) λ_{11} is larger than λ_{12}, irrespective of the latter's sign.

Example: Theory 2

Bass (1999) Theory

$$\mathbf{\Lambda}_2 = \begin{bmatrix} \lambda_{11} & > & |\lambda_{12}| \\ \lambda_{21} > 0 & & \lambda_{22} = 0 \\ \lambda_{31} < -.3 & & \lambda_{32} > .3 \\ \lambda_{41} & > & |\lambda_{42}| \\ \lambda_{51} & > & |\lambda_{52}| \\ \lambda_{61} = 0 & & \lambda_{62} > 0 \end{bmatrix} \begin{matrix} \text{IL} \\ \text{ROL} \\ \text{PL} \\ \text{RML} \\ \text{IFL} \\ \text{UL} \end{matrix},$$

Social exchange vs. laissez-faire leadership styles.

Example: Theory 3

Treviño (1986) Theory

$$\mathbf{\Lambda}_3 = \begin{bmatrix} |\lambda_{11}| & < & -\lambda_{12} \\ \lambda_{21} > 0 & & \lambda_{22} = 0 \\ |\lambda_{31}| & < & \lambda_{32} \\ |\lambda_{41}| & < & -\lambda_{42} \\ \lambda_{51} & > & -\lambda_{52} \\ \lambda_{61} = 0 & & \lambda_{62} > 0 \end{bmatrix} \begin{matrix} \text{IL} \\ \text{ROL} \\ \text{PL} \\ \text{RML} \\ \text{IFL} \\ \text{UL} \end{matrix}.$$

Reinforcement leadership vs. the full ethical range.

Example: Results

Table: Estimated Bayes factors for constrained models on the leadership data

M_1	M_2	M_3
$\hat{f}_1^F = .000$	$\hat{f}_2^F = .533$	$\hat{f}_3^F = .000$
$\omega_1^F = 2^{-8}$	$\hat{\omega}_2^F = 1.518\mathrm{e}-3$	$\omega_3^F = 2^{-7}$
$\hat{B}_{10}^F = .000$	$\hat{B}_{20}^F = 350.993$	$\hat{B}_{30}^F = .000$

Table: Bayes factor matrix for the leadership data

$\hat{\mathrm{B}}_{from\|to}$	0	1	2	3
0	1	∞	$2.849\mathrm{e}-3$	∞
1	.000	1	.000	–
2	350.993	∞	1	∞
3	.000	–	.000	1

Outline

Remarks

All is Good

- Flexible way of performing model selection
- Allows for more specificity in model-translated theories
- Allows for direct testing competing theories

All is Relative

- Computational intensity

Outline

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Outline

Discussion

Thank you for listening. All questions and points of discussion welcome.

Outline

Proposition 1

Proposition

- Define $\mathbf{\Theta}_b \equiv \{\mathbf{\Theta} : \mathbf{\Omega}_b\boldsymbol{\theta}^{\mathfrak{c}} - \boldsymbol{\alpha}_b > \mathbf{0}\}$
- Let the prior under encompassing model M_0 for the set of parameters in the model be $\pi_0(\mathbf{\Theta})g(\boldsymbol{\vartheta}|\boldsymbol{\theta}^{\mathfrak{u}})$
- The prior distribution under any constrained model M_b is $\pi_b(\mathbf{\Theta}_b)g(\boldsymbol{\vartheta}|\boldsymbol{\theta}^{\mathfrak{u}}) \propto \pi_0(\mathbf{\Theta})\mathbf{1}_{\{\mathbf{\Omega}_b\boldsymbol{\theta}^{\mathfrak{c}} - \boldsymbol{\alpha}_b > \mathbf{0}\}}g(\boldsymbol{\vartheta}|\boldsymbol{\theta}^{\mathfrak{u}})$
- Assume propriety of $\pi_0(\mathbf{\Theta}, \boldsymbol{\vartheta}|\mathbf{X})$

The Bayes factor B_{b0} reduces to the posterior probability mass satisfying the system of constraints that defines $\mathbf{\Theta}_b$ over the prior probability mass satisfying the system of constraints that defines $\mathbf{\Theta}_b$:

$$B_{b0} = \frac{\int\int_{\{\mathbf{\Theta}:\mathbf{\Omega}_b\boldsymbol{\theta}^{\mathfrak{c}} - \boldsymbol{\alpha}_b > \mathbf{0}\}} \pi_0(\mathbf{\Theta}, \boldsymbol{\vartheta}|\mathbf{X})\,\partial\mathbf{\Theta}\,\partial\boldsymbol{\vartheta}}{\int_{\{\mathbf{\Theta}:\mathbf{\Omega}_b\boldsymbol{\theta}^{\mathfrak{c}} - \boldsymbol{\alpha}_b > \mathbf{0}\}} \pi_0(\mathbf{\Theta})\,\partial\mathbf{\Theta}} \equiv \frac{f_b}{\omega_b}, \tag{1}$$

Proof Proposition 1

The Bayes factor B_{b0} of constrained model M_b to encompassing model M_0, is written as

$$\frac{\int L(\boldsymbol{\mu}, \boldsymbol{\Lambda}_0, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi}; \mathbf{Z})\pi(\boldsymbol{\mu}, \boldsymbol{\Psi})\pi_b(\boldsymbol{\Lambda}_b|\boldsymbol{\Phi})g(\boldsymbol{\Xi}|\boldsymbol{\Phi})\pi(\boldsymbol{\Phi})\,\partial(\boldsymbol{\mu}, \boldsymbol{\Lambda}_b, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi})}{\int L(\boldsymbol{\mu}, \boldsymbol{\Lambda}_0, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi}; \mathbf{Z})\pi(\boldsymbol{\mu}, \boldsymbol{\Psi})\pi_0(\boldsymbol{\Lambda}_0|\boldsymbol{\Phi})g(\boldsymbol{\Xi}|\boldsymbol{\Phi})\pi(\boldsymbol{\Phi})\,\partial(\boldsymbol{\mu}, \boldsymbol{\Lambda}_0, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi})}. \quad (2)$$

Using the basic marginal identity we may express (2) equivalently as follows:

$$\frac{L(\boldsymbol{\mu}, \boldsymbol{\Lambda}_0, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi}; \mathbf{Z})\pi(\boldsymbol{\mu}, \boldsymbol{\Psi})\pi_b(\boldsymbol{\Lambda}_b|\boldsymbol{\Phi})g(\boldsymbol{\Xi}|\boldsymbol{\Phi})\pi(\boldsymbol{\Phi})/\pi_b(\boldsymbol{\mu}, \boldsymbol{\Lambda}_b, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi}|\mathbf{Z})}{L(\boldsymbol{\mu}, \boldsymbol{\Lambda}_0, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi}; \mathbf{Z})\pi(\boldsymbol{\mu}, \boldsymbol{\Psi})\pi_0(\boldsymbol{\Lambda}_0|\boldsymbol{\Phi})g(\boldsymbol{\Xi}|\boldsymbol{\Phi})\pi(\boldsymbol{\Phi})/\pi_0(\boldsymbol{\mu}, \boldsymbol{\Lambda}_0, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi}|\mathbf{Z})}.$$

Proof Proposition 1

For any given value of $\{\boldsymbol{\mu}, \boldsymbol{\Lambda}_0, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi}\}$, say $\{\boldsymbol{\mu}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Xi}^*, \boldsymbol{\Psi}^*, \boldsymbol{\Phi}^*\}$, that is admissible under the system of constraints $\boldsymbol{\Omega}_b \boldsymbol{\lambda}_f - \boldsymbol{\alpha}_b > \mathbf{0}$, clearly $\{\boldsymbol{\mu}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Xi}^*, \boldsymbol{\Psi}^*, \boldsymbol{\Phi}^*\} \in M_b \cap M_0$. For any such value we then have:

$$\frac{L(\boldsymbol{\mu}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Xi}^*, \boldsymbol{\Psi}^*, \boldsymbol{\Phi}^*; \mathbf{Z})\pi(\boldsymbol{\mu}^*, \boldsymbol{\Psi}^*)\pi_b(\boldsymbol{\Lambda}^*|\boldsymbol{\Phi}^*)g(\boldsymbol{\Xi}^*|\boldsymbol{\Phi}^*)\pi(\boldsymbol{\Phi}^*)}{L(\boldsymbol{\mu}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Xi}^*, \boldsymbol{\Psi}^*, \boldsymbol{\Phi}^*; \mathbf{Z})\pi(\boldsymbol{\mu}^*, \boldsymbol{\Psi}^*)\pi_0(\boldsymbol{\Lambda}^*|\boldsymbol{\Phi}^*)g(\boldsymbol{\Xi}^*|\boldsymbol{\Phi}^*)\pi(\boldsymbol{\Phi}^*)} \cdot \frac{\pi_0(\boldsymbol{\mu}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Xi}^*, \boldsymbol{\Psi}^*, \boldsymbol{\Phi}^*|\mathbf{Z})}{\pi_b(\boldsymbol{\mu}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Xi}^*, \boldsymbol{\Psi}^*, \boldsymbol{\Phi}^*|\mathbf{Z})}. \tag{3}$$

Dividing out terms, expression (3) reduces to

$$\frac{\pi_0(\boldsymbol{\mu}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Xi}^*, \boldsymbol{\Psi}^*, \boldsymbol{\Phi}^*|\mathbf{Z})}{\pi_b(\boldsymbol{\mu}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Xi}^*, \boldsymbol{\Psi}^*, \boldsymbol{\Phi}^*|\mathbf{Z})} \cdot \frac{\pi_b(\boldsymbol{\Lambda}^*|\boldsymbol{\Phi}^*)}{\pi_0(\boldsymbol{\Lambda}^*|\boldsymbol{\Phi}^*)}. \tag{4}$$

Proof Proposition 1

Now, notice

$$\begin{aligned}
&\pi_b(\boldsymbol{\mu}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Xi}^*, \boldsymbol{\Psi}^*, \boldsymbol{\Phi}^* | \mathbf{Z}) \\
&\quad = \pi_0(\boldsymbol{\mu}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Xi}^*, \boldsymbol{\Psi}^*, \boldsymbol{\Phi}^* | \mathbf{Z}) \\
&\qquad \cdot \left[\int\int_{\{\boldsymbol{\Lambda}_0 : \boldsymbol{\Omega}_b \boldsymbol{\lambda}_f - \boldsymbol{\alpha}_b > \mathbf{0}\}} \pi_0(\boldsymbol{\mu}, \boldsymbol{\Lambda}_0, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi} | \mathbf{Z}) \, \partial \boldsymbol{\Lambda}_0 \, \partial(\boldsymbol{\mu}, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi}) \right]^{-1}, \quad (5)
\end{aligned}$$

and

$$\pi_b(\boldsymbol{\Lambda}^* | \boldsymbol{\Phi}^*) = \pi_0(\boldsymbol{\Lambda}^* | \boldsymbol{\Phi}^*) \cdot \left[\int_{\{\boldsymbol{\Lambda}_0 : \boldsymbol{\Omega}_b \boldsymbol{\lambda}_f - \boldsymbol{\alpha}_b > \mathbf{0}\}} \pi_0(\boldsymbol{\Lambda}_0 | \boldsymbol{\Phi}) \, \partial \boldsymbol{\Lambda}_0 \right]^{-1}. \qquad (6)$$

Substituting (5) and (6) in (4) we obtain

$$B_{b0} = \frac{\int\int_{\{\boldsymbol{\Lambda}_0 : \boldsymbol{\Omega}_b \boldsymbol{\lambda}_f - \boldsymbol{\alpha}_b > \mathbf{0}\}} \pi_0(\boldsymbol{\mu}, \boldsymbol{\Lambda}_0, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi} | \mathbf{Z}) \, \partial \boldsymbol{\Lambda}_0 \, \partial(\boldsymbol{\mu}, \boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi})}{\int_{\{\boldsymbol{\Lambda}_0 : \boldsymbol{\Omega}_b \boldsymbol{\lambda}_f - \boldsymbol{\alpha}_b > \mathbf{0}\}} \pi_0(\boldsymbol{\Lambda}_0 | \boldsymbol{\Phi}) \, \partial \boldsymbol{\Lambda}_0} = \frac{f_b}{\omega_b}, \qquad (7)$$

and the proposition follows.

Priors

Total prior density

$$\pi(\boldsymbol{\mu}, \boldsymbol{\Psi})\pi(\boldsymbol{\Lambda}_0|\boldsymbol{\Phi})g(\boldsymbol{\Xi}|\boldsymbol{\Phi})\pi(\boldsymbol{\Phi}) = \pi(\boldsymbol{\mu})\pi(\boldsymbol{\Psi})\pi(\boldsymbol{\Lambda}_0)\mathbf{1}_{\{\mathbf{I}_p - \text{diag}(\boldsymbol{\Lambda}_0\boldsymbol{\Phi}\boldsymbol{\Lambda}_0^{\mathrm{T}})>0\}} g(\boldsymbol{\Xi}|\boldsymbol{\Phi})\pi(\boldsymbol{\Phi}).$$

Individual priors

$$\begin{aligned}
\pi(\mu_j) &\propto 1 \quad \mathcal{L}(-\infty, \infty) \\
\pi(\lambda_{jk}) &\propto 1 \quad \mathcal{L}(-1, 1) \\
\pi((-)\lambda_{jk}^{p}) &\propto 1 \quad \mathcal{L}(0, 1) \\
\psi_{jj} &\sim \mathcal{IG}(\nu/2, \nu\kappa/2) \\
\boldsymbol{\xi}_i &\sim \mathcal{N}_m(\mathbf{0}, \boldsymbol{\Phi}) \\
\pi(\boldsymbol{\Phi}) &\sim \mathcal{IW}_m(\boldsymbol{\Upsilon}, \tau)
\end{aligned}$$

Conditionals

$$\pi(\boldsymbol{\mu}|\mathbf{Z}, \boldsymbol{\Lambda}_0, \boldsymbol{\Xi}, \boldsymbol{\Psi}) \overset{d}{=} \mathcal{N}_p(\tilde{\boldsymbol{\mu}}, \boldsymbol{\Sigma}_{\boldsymbol{\mu}})$$

$$\pi(\boldsymbol{\Lambda}_{0j}^{*\mathrm{T}}|\mathbf{z}_j, \mu_j, \boldsymbol{\Xi}_j^*, \psi_{jj}, \boldsymbol{\Phi}^s) \overset{d}{=} \mathcal{N}_m(\tilde{\boldsymbol{\Lambda}}_{0j}^*, \boldsymbol{\Sigma}_{\boldsymbol{\Lambda}_{0j}}^*)$$

$$\mathbf{1}_{\left\{0<(-)\lambda_{jk}^p<1 \,\cap\, -1<\lambda_{jk'}<1 \;\forall k'\neq k \,\cap\, 1-\boldsymbol{\Lambda}_{0j}\boldsymbol{\Phi}^s\boldsymbol{\Lambda}_{0j}^{\mathrm{T}}>0\right\}}$$

$$\pi(\boldsymbol{\Lambda}_{0j}^{*\mathrm{T}}|\mathbf{z}_j, \mu_j, \boldsymbol{\Xi}_j^*, \psi_{jj}, \boldsymbol{\Phi}^s) \overset{d}{=} \mathcal{N}_m(\tilde{\boldsymbol{\Lambda}}_{0j}^*, \boldsymbol{\Sigma}_{\boldsymbol{\Lambda}_{0j}}^*)\mathbf{1}_{\left\{-1<\lambda_{jk'}<1 \;\forall k \,\cap\, 1-\boldsymbol{\Lambda}_{0j}\boldsymbol{\Phi}^s\boldsymbol{\Lambda}_{0j}^{\mathrm{T}}>0\right\}}$$

$$\pi(\boldsymbol{\xi}_i|\mathbf{z}_i, \boldsymbol{\mu}, \boldsymbol{\Lambda}_0, \boldsymbol{\Psi}, \boldsymbol{\Phi}^s) \overset{d}{=} \mathcal{N}_m(\tilde{\boldsymbol{\xi}}, \boldsymbol{\Sigma}_\xi)$$

$$\pi(\boldsymbol{\Phi}|\mathbf{Z}, \boldsymbol{\Xi}) \overset{d}{=} \mathcal{IW}_m(\boldsymbol{\Xi}^{\mathrm{T}}\boldsymbol{\Xi} + \boldsymbol{\Upsilon}, n+\tau)$$

$$\pi(\psi_{jj}|\mathbf{z}_j, \mu_j, \boldsymbol{\Lambda}_{0j}, \boldsymbol{\Xi}) \overset{d}{=} \mathcal{IG}((n+\nu)/2, (\beta_j + \nu\kappa)/2)$$

$$\phi_{kk'}^s = \phi_{kk'}/\sqrt{\phi_{kk}} \cdot \sqrt{\phi_{k'k'}}$$

Complexity

Equation (7) states that the complexity of an inequality constrained factor model is given by

$$\int_{\{\boldsymbol{\Lambda}_0 : \boldsymbol{\Omega}_b \boldsymbol{\lambda}_f - \boldsymbol{\alpha}_b > \mathbf{0}\}} \pi_0(\boldsymbol{\Lambda}_0 | \boldsymbol{\Phi}) \, \partial \boldsymbol{\Lambda}_0$$
$$= \int_{\{\boldsymbol{\Lambda}_0 : \boldsymbol{\Omega}_b \boldsymbol{\lambda}_f - \boldsymbol{\alpha}_b > \mathbf{0}\}} \pi_0(\boldsymbol{\Lambda}_0) \mathbf{1}_{\{\mathbf{I}_p - \mathrm{diag}(\boldsymbol{\Lambda}_0 \boldsymbol{\Phi} \boldsymbol{\Lambda}_0^{\mathrm{T}}) > 0\}} \, \partial \boldsymbol{\Lambda}_0, \quad (8)$$

indicating the proportion of the domain of M_0 abiding the inequality constraints that define M_b. In this form, $\boldsymbol{\Phi}$ is unknown. Notice, however, that for any positive definite $\boldsymbol{\Phi}$ we may always find $\mathbf{V} \in \mathbb{R}^{m \times m}$ such that $\boldsymbol{\Phi} = \mathbf{V}\mathbf{V}^{\mathrm{T}}$, and $\boldsymbol{\Lambda}_0 \boldsymbol{\Phi} \boldsymbol{\Lambda}_0^{\mathrm{T}} = (\boldsymbol{\Lambda}_0 \mathbf{V})(\boldsymbol{\Lambda}_0 \mathbf{V})^{\mathrm{T}}$. This implies that each oblique representation has equivalent orthogonal representations and suggests evaluation of complexity under orthogonality. This coincides with (integrating $\boldsymbol{\Phi}$ out of $\pi_0(\boldsymbol{\Lambda}_0, \boldsymbol{\Phi})$ and) evaluating $\boldsymbol{\Phi}$ in $\mathbf{1}_{\{\mathbf{I}_p - \mathrm{diag}(\boldsymbol{\Lambda}_0 \boldsymbol{\Phi} \boldsymbol{\Lambda}_0^{\mathrm{T}}) > 0\}}$ at its prior expectation: $\boldsymbol{\Upsilon}(\tau - m - 1)^{-1} = \mathbf{I}_m$. Model complexity is thus formalized as follows:

Definition Complexity

Let $\dot{\mathbf{\Lambda}}_{0j}^{\star}$ denote the $(m - t_j)$-dimensional row vector containing the elements in $\mathbf{\Lambda}_{0j}$ not involved in restrictions under conditions *C1* and *C4*. Let the uniform density (conform prior specification) on $\dot{\mathbf{\Lambda}}_{0j}^{\star}$ be given by:

$$\varrho(\dot{\mathbf{\Lambda}}_{0j}^{\star}; -1, 1) = \begin{cases} (2)^{-(m-t_j)} & \text{if } \dot{\mathbf{\Lambda}}_{0j}^{\star} \in (-1, 1)^{m-t_j} \\ 0 & \text{elsewhere} \end{cases} .$$

Moreover, let

$$\varrho((-)\lambda_{jk}^{p}; 0, 1) = \begin{cases} 1 & \text{if } (-)\lambda_{jk}^{p} \in (0, 1) \\ 0 & \text{elsewhere} \end{cases} ,$$

denote the uniform density on the factor loading elements involved in a polarity truncation as indicated by condition *C4* and define

$$\Gamma_j = \begin{cases} \varrho((-)\lambda_{jk}^{p}; 0, 1) & \text{if } \lambda_{jk}^{p} \in \mathbf{\Lambda}_{0j}^{\star} \\ 1 & \text{otherwise} \end{cases} .$$

Definition Complexity

Model complexity ω_b of an inequality constrained model M_b is then defined as the probability mass of the j-product of $\varrho(\dot{\mathbf{\Lambda}}^{\star}_{0j}; -1, 1)\Gamma_j$, bounded by the correlation restriction $1 - \mathbf{\Lambda}_{0j}\mathbf{\Phi}\mathbf{\Lambda}_{0j}^{\mathrm{T}} > 0$ evaluated at the prior expectation of $\mathbf{\Phi}$, that is located in the inequality constrained space defining M_b. Hence,

$$\omega_b \equiv \int_{\{\mathbf{\Lambda}_0 : \mathbf{\Omega}_b \boldsymbol{\lambda}_f - \mathbf{c}_b > \mathbf{0}\}} \prod_{j=1}^{p} \varrho(\dot{\mathbf{\Lambda}}^{\star}_{0j}; -1, 1)\Gamma_j \mathbf{1}_{\left\{1 - \mathbf{\Lambda}_{0j}\mathbf{\Lambda}_{0j}^{\mathrm{T}} > 0\right\}} \partial \mathbf{\Lambda}_0. \tag{9}$$

Gibbs Sampler for $\hat{f}_b$

1: Set $\mathbf{\Lambda}_0^{(0)}$, $\mathbf{\Xi}^{(0)}$, $\mathbf{\Phi}^{s(0)}$, and $\mathbf{\Psi}^{(0)}$
2: **for** $c = 1$ to C **do**
3: Generate $\boldsymbol{\mu}^{(c)}$ from $\pi(\boldsymbol{\mu}|\mathbf{Z}, \mathbf{\Lambda}_0^{(c-1)}, \mathbf{\Xi}^{(c-1)}, \mathbf{\Psi}^{(c-1)})$
4: Generate $\mathbf{\Lambda}_0^{(c)}$ from $\prod_{j=1}^{p} \zeta(\tilde{\mathbf{\Lambda}}_{0j}^{\star}, \mathbf{\Sigma}_{\mathbf{\Lambda}_{0j}^{\star}})$
5: **if** $\exists jk$ such that $\neg(-1 < \lambda_{jk}^{(c)} < 1)$ or $\neg(0 < (-)\lambda_{jk}^{p(c)} < 1)$ **or** $\mathbf{I}_p - \mathrm{diag}\left(\mathbf{\Lambda}_0^{(c)} \mathbf{\Phi}^{s(c-1)} \mathbf{\Lambda}_0^{(c)\mathrm{T}}\right) \not> 0$ **then**
6: go to 4:
7: **else**
8: Generate $\mathbf{\Xi}^{(c)}$ from $\prod_{i=1}^{n} \pi(\boldsymbol{\xi}_i|\mathbf{z}_i, \boldsymbol{\mu}^{(c)}, \mathbf{\Lambda}_0^{(c)}, \mathbf{\Psi}^{(c-1)}, \mathbf{\Phi}^{s(c-1)})$
9: Generate $\mathbf{\Phi}^{(c)}$ from $\pi(\mathbf{\Phi}|\mathbf{Z}, \mathbf{\Xi}^{(c)})$
10: Set $\forall k \geqslant k'$ $\phi_{kk'}^{s(c)} = \phi_{kk'}^{(c)} / \sqrt{\phi_{kk}^{(c)}} \cdot \sqrt{\phi_{k'k'}^{(c)}}$
11: Set $\forall k > k'$ $\phi_{k'k}^{s(c)} = \phi_{kk'}^{s(c)}$
12: Generate $\mathbf{\Psi}^{(c)}$ from $\prod_{j=1}^{p} \pi(\psi_{jj}|\mathbf{z}_j, \mu_j^{(c)}, \mathbf{\Lambda}_{0j}^{(c)}, \mathbf{\Xi}^{(c)})$
13: **end if**
14: **end for**
15: $\hat{f}_b = C^{-1} \sum_{c=1}^{C} \mathbf{1}_{\{\mathbf{\Omega}_b \boldsymbol{\lambda}_f^{(c)} - \boldsymbol{\alpha}_b > \mathbf{0}\}}$

Sampler for $\hat{\omega}_b$

1: Set $\varphi = 0$

2: **for** $v = 1$ to V **do**

3: Generate $\boldsymbol{\Lambda}_0^{(v)}$ from $\prod_{j=1}^{p} \varrho(\dot{\boldsymbol{\Lambda}}_{0j}^{\star}; -1, 1)\Gamma_j$

4: **if** $\mathbf{I}_p - \mathrm{diag}\left(\boldsymbol{\Lambda}_0^{(v)}\boldsymbol{\Lambda}_0^{(v)\mathrm{T}}\right) \not\succ 0$ **then**

5: go to 3:

6: **else**

7: **if** $\mathbf{1}_{\{\boldsymbol{\Omega}_b\boldsymbol{\lambda}_f^{(v)} - \boldsymbol{\alpha}_b > \mathbf{0}\}} = 1$ **then**

8: $\varphi = \varphi + 1$

9: **else**

10: $\varphi = \varphi$

11: **end if**

12: **end if**

13: **end for**

14: $\hat{\omega}_b = V^{-1}\varphi$